SOME THEOREMS ON L^p FOURIER SERIES

BY

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1. The almost everywhere convergence of lacunary sequences of partial sums has long been known for L^2 Fourier series [8, p. 251]. Somewhat later the result was established for class L^p by Littlewood and Paley [4] and for class H by Zygmund, [6] and [7]. In the papers of Zygmund, he also extended to class H an unpublished result of Paley concerning L^p series. This stated that for f in L^p, $p>1$, and for almost every x, the positive integers can be divided into complementary sequences, $\{m_\nu\}$ and $\{n_\nu\}$, in general depending on x, such that $s_{m_\nu}(x;f)$ converges to $f(x)$ and such that $\sum 1/n_\nu < \infty$. Here $s_m(x;f)$ denotes as usual the mth partial sum of the Fourier series of f at x. The sequence $\{m_\nu\}$ depends then on both the function f and the point x. Results intermediate between these have recently appeared [1] in which the sequence of indices of the partial sums for which convergence takes place are more dense in a certain sense than lacunary sequences, less dense than those described immediately above; and they depend on the function f but not on the point x. In particular, for each f in L^2, there is a sequence $\{m_\nu\}$ of upper density one such that $s_{m_\nu}(x;f)$ converges to f almost everywhere.

In this paper we shall generalize our initial results on L^p series, which were largely based on the Hausdorff-Young theorem. Much stronger results will be obtained by the use of powerful theorems of Littlewood and Paley. In the next section we give our preliminary lemmas; in the third section our two main theorems; and in the final section additional results which are mainly corollaries of our main results.

2. Our theorems depend to a large extent on a certain result of Littlewood and Paley [4, II], which we now describe. Let $\{m_k\}$ and $\{n_k\}$ be two sequences of positive integers satisfying $1<\alpha\leqq m_{k+1}/m_k\leqq\beta$, $1<\alpha\leqq n_{k+1}/n_k\leqq\beta$ for fixed α, β. Let f belong to L^p, $p>1$, and let $\sum_{m=-\infty}^{+\infty} c_m e^{imx}$ be its Fourier series. We may assume throughout that $c_0=0$. For $k>0$, let $\Delta_k(x) = \sum_{\nu=n_{k-1}+1}^{n_k} c_\nu e^{i\nu x}$; $\Delta_{-k}(x)=\sum_{\nu=-m_{k-1}-1}^{-m_k} c_\nu e^{i\nu x}$ if it is understood that $m_0=n_0=0$. The theorem states that

$$(1)\quad A_{p,\alpha,\beta}\int_0^{2\pi} |f(x)|^p dx \leqq \int_0^{2\pi}\left(\sum_{k=-\infty}^{+\infty} |\Delta_k(x)|^2\right)^{p/2} dx \leqq B_{p,\alpha,\beta}\int_0^{2\pi} |f(x)|^p dx$$

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where $A_{p,\alpha,\beta}$, $B_{p,\alpha,\beta}$ are constants depending only on their subscripts.

In general it will be sufficient to consider only those f such that $c_n=0$ for $n\leqq 0$; and we do this now. Letting $n_k=\lambda^k$ for λ an integer exceeding 1, we have that the constants A and B in (1) depend on p and λ. We shall be interested in blocks of consecutive $\Delta_k(x)$'s. Thus, for $K_2>K_1$ by (1)

$$(2)\qquad \int_0^{2\pi}\left(\sum_{k=K_1+1}^{K_2} |\Delta_k(x)|^2\right)^{p/2} dx \leqq B_{\lambda,p}\int_0^{2\pi} |s_{\lambda^{K_2}}(x;f)-s_{\lambda^{K_1}}(x;f)|^p dx.$$

The constant $B_{\lambda,p}$ may be large, but by the well known fact of the mean convergence of L^p Fourier series, the right side of (2) may be made arbitrarily small for fixed λ by taking K_1 and K_2 sufficiently large. We let

$$\mathcal{E}_k=\int_0^{2\pi} |\Delta_k(x)|^p dx,\qquad \delta=\int_0^{2\pi}\left(\sum_{k=K}^{2K-1} |\Delta_k(x)|^2\right)^{p/2} dx.$$

For a real number y, let $\langle y\rangle$ denote the greatest integer not exceeding y. The principle involved in our first lemma will be useful in all our proofs.

LEMMA 1. *For $\beta>0$, among the K numbers $\mathcal{E}_k$, no more than $\langle K/\beta\rangle$ of them exceed $\beta\delta K^{-p/2}$.*

The proof follows directly from Hölder's inequality which gives

$$\sum_{k=K}^{2K-1} |\Delta_k(x)|^p \leqq K^{1-p/2}\left(\sum_{k=K}^{2K-1} |\Delta_k(x)|^2\right)^{p/2}.$$

Integration of this inequality gives

$$\sum_{k=K}^{2K-1}\mathcal{E}_k \leqq K^{1-p/2}\delta.$$

If N of the numbers $\mathcal{E}_k$ exceed $\beta\delta K^{-p/2}$, then from the above $N\beta\delta K^{-p/2}\leqq K^{1-p/2}\delta$; or $N\leqq K/\beta$ as required.

We shall also need another result of Littlewood and Paley [4, III].

LEMMA 2. *If f belongs to L^2, then*

$$\int_0^{2\pi}\sup_n\left\{\frac{|s_n(x;f)|^2}{\log(n+2)}\right\}dx \leqq A_2\int_0^{2\pi}|f(x)|^2dx.$$

The original proof covers the L^p case, $1<p\leqq 2$, and is complicated. A relatively simple proof covers the L^2 case and will be found in [2]. The L analogue of Lemma 2 is false [7] although it is well known, for example, that almost everywhere $s_n(x;f)=o(\log n)$ [8, p. 32]. However we shall require the somewhat different result of the following lemma, which involves the function

$$s_n^*(x;f) = \sup_{m \leqq n} | s_m(x;f) |.$$

LEMMA 3. *Given $M>0$ and f in L, let E_M be the set of x values in $[0, 2\pi]$ for which $s_n^*(x; f) > M \log(n+2)$. Then E_M has measure not exceeding $AM^{-1}\int_0^{2\pi} f(x)dx$, where A is a constant.*

Let $F=\int_0^{2\pi}|f(t)|dt$. If $F/M>2\pi(n+2)$, then the condition on $|E_M|$, the measure of E_M, is automatically satisfied with $A\geqq 1$ since $|E_M|\leqq 2\pi$. If, on the other hand, $2\pi(n+2)F<M$, then E_M has measure 0 since

$$| s_m(x;f) | \leqq \frac{1}{\pi}\int_0^{2\pi} | f(t) | \, | D_m(x-t) | \, dt \leqq (n+1)\frac{F}{\pi}, \qquad m \leqq n$$

where $D_m(x)$ is the Dirichlet kernel. Hence we may assume that

$$(2\pi(n+2))^{-1} \leqq FM^{-1} \leqq 2\pi(n+2).$$

Let Q be the least integer exceeding both MF^{-1} and $4(n+2)FM^{-1}$ so that $1\leqq Q\leqq 8\pi(n+2)^2+1$. Let I_ν be the interval $2\pi\nu Q^{-2}\leqq x\leqq 2\pi(\nu+1)Q^{-2}$, $\nu=0, 1, \cdots, Q^2-1$. Then, except possibly for Q values of ν,

$$\int_{I_\nu} |f(t)|dt \leqq FQ^{-1}.$$

Let G_1 be the exceptional set of I_ν's. Then, $|G_1|\leqq 2\pi Q^{-1}$. Let G_2 be the set of intervals of length $2\pi Q^{-3}$ situated symmetrically about the points $2\pi\nu Q^{-2}$, $\nu=0, 1, \cdots, Q^2-1$. Then $|G_2|\leqq 2\pi Q^{-1}$. If x belongs to the complement of $G_1\cup G_2$ and also belongs to I_ν, then

$$\int_{0\leqq|x-t|\leqq\pi/Q^3} | f(t) | \, dt \leqq \int_{I_\nu} | f(t) | \, dt \leqq FQ^{-1}.$$

For $m\leqq n$,

$$\begin{aligned} | s_m(x;f) | &\leqq \frac{(n+1/2)}{\pi}\int_{0\leqq|x-t|\leqq\pi/Q^3} | f(t) | \, dt + \frac{1}{2}\int_{\pi/Q^3\leqq|x-t|\leqq\pi} \frac{| f(t) |}{| x-t |}\, dt \\ &= T_1(x) + T_2(x). \end{aligned}$$

Since $\int_0^{2\pi} T_2(x)dx\leqq 2F(\log Q^3)$, then $2T_2(x)\leqq M\log(n+2)$ for x outside a set G_3 of measure not exceeding $12F(\log Q)/M\log(n+2)$. For x outside $G_1\cup G_2$

$$T_1(x) \leqq \frac{n+1/2}{\pi}\,\frac{F}{Q} \leqq \frac{M}{4} < \frac{M}{2}\log(n+2).$$

Thus for x in the complement of $G_1\cup G_2\cup G_3$, $s_n^*(x;f)\leqq M\log(n+2)$. The measure of this set does not exceed

$$\frac{4\pi}{Q} + \frac{12F\log Q}{M\log(n+2)} \leqq \frac{4\pi F}{M} + \frac{12F}{M\log(n+2)}\log\left[8\pi(n+2)^2+1\right] \leqq \frac{CF}{M}$$

for some constant C.

Lemma 3 leads in an easy way to a weak analogue of Lemma 2; thus if f belongs to L,

$$\int_0^{2\pi} [s_n^*(x;f)]^r dx \leqq C_r[\log(n+2)]^r \left[\int_0^{2\pi} |f(x)|\, dx\right]^r, \qquad 0 < r < 1.$$

To prove this we may assume that $s_n^*(x;f) \geqq F\log(n+2)$. Let G_m be the set of x values for which $s_n^*(x;f) \geqq Fm\log(n+2)$. By Lemma 3, $|G_m| \leqq Am^{-1}$. Then our result follows from the sequence of inequalities

$$\int_0^{2\pi} [s_n^*(x;f)]^r dx \leqq [\log(n+2)]^r F^r \sum_{m=1}^{\infty} (m+1)^r [|G_m| - |G_{m+1}|]$$

$$\leqq [2\log(n+2)]^r F^r \sum_{m=1}^{\infty} m^{r-1} |G_m| \leqq A[2\log(n+2)]^r F^r \sum_{m=1}^{\infty} m^{r-2}.$$

3. Our first theorem deals with the density function for a sequence of distinct positive integers: given $\{m_\nu\}$, let $\sigma(n)$ be the number of terms of the sequence not exceeding n. For the L^2 case we were able to prove that $s_{m_\nu}(x;f)$ converged to f almost everywhere for some sequence of upper density one: i.e. such that $\limsup \sigma(n)/n = 1$ [1, p. 396]. Here the corresponding result for L^p, $1<p<2$, is not the same, but it merges, so to speak, with the L^2 result. We shall say that a sequence $\{m_\nu\}$ satisfies condition C_p if

$$(C_p) \qquad \limsup_{n\to\infty} \frac{(\log n)^{(2-p)/2(p-1)}\sigma(n)}{n} \geqq 1.$$

THEOREM 1. *If f belongs to L^p, $1<p\leqq 2$, there is a sequence $\{m_\nu\}$ satisfying (C_p) such that $s_{m_\nu}(x;f)$ converges to f almost everywhere.*

We assume first that $c_n=0$ for $n\leqq 0$. Let $\{\lambda_r\}$ and $\{k_r\}$ be sequences of positive integers, in general large; and let $n_k=\lambda_r^k$, $k=k_r, k_r+1, \cdots, 2k_r$. Let

$$\int_0^{2\pi} \left(\sum_{k=k_r}^{2k_r-1} |\Delta_k(x)|^2\right)^{p/2} dx = \delta_r, \qquad \Delta_k(x) = \sum_{n=n_k+1}^{n_{k+1}} c_n e^{inx}.$$

As noted in the previous section, δ_r depends not only on f but also on λ_r and k_r; but it can be made as small as we please, given λ_r, by proper choice of k_r. Let E_r be the set of x values in $[0, 2\pi]$ for which

$$\sum_{k=k_r}^{2k_r-1} |\Delta_k(x)|^2 > (k_r)^{1/(p-1)}.$$

From the preceding, $|E_r|$, the measure of E_r, does not exceed $\delta_r k_r^{-q/2}$ where q is the conjugate of p, i.e. $p^{-1}+q^{-1}=1$. Let $\gamma=(2-p)/2(p-1)$. On E_r', the complement of E_r,

$$\sum_{k=k_r}^{2k_r-1} |\Delta_k(x)|^2 \leqq k_r^{\gamma}\left(\sum_{k=k_r}^{2k_r-1} |\Delta_k(x)|^2\right)^{p/2}.$$

Integrating this inequality over E_r', we obtain

$$\sum_{k=k_r}^{2k_r-1} \int_{E_r'} |\Delta_k(x)|^2 dx \leqq k_r^{\gamma}\delta_r.$$

By the same reasoning as that used in the proof of Lemma 1, no more than $\langle k_r/3\rangle$ of the numbers $\int_{E_r'}|\Delta_k(x)|^2dx$, $k=k_r, \cdots, 2k_r-1$, exceed $3k_r^{\gamma-1}\delta_r$. By Lemma 1 itself, no more than $\langle k_r/3\rangle$ of the numbers $\int_0^{2\pi}|\Delta_k(x)|^pdx$, $k=k_r, \cdots, 2k_r-1$, exceed $3k_r^{-p/2}\delta_r$. Hence, for at least one k, say $k(r)$ satisfying $k_r \leqq k(r) \leqq 2k_r-1$, the following simultaneous inequalities hold.

$$(3) \qquad \int_{E_r'} |\Delta_{k(r)}(x)|^2 dx \leqq 3k_r^{\gamma-1}\delta_r, \qquad \int_0^{2\pi} |\Delta_{k(r)}(x)|^p dx \leqq 3k_r^{-p/2}\delta_r.$$

Let $\mathfrak{X}$ be the characteristic function of the set E_r. To investigate the partial sums of the Fourier series of $\Delta_{k(r)}$, we consider separately the partial sums of the Fourier series of the two functions $\mathfrak{X}\Delta_{k(r)}$ and $(1-\mathfrak{X})\Delta_{k(r)}$. By Holder's inequality and (3) above

$$\int_0^{2\pi} |\mathfrak{X}(x)\Delta_{k(r)}(x)| dx \leqq \left(\int_0^{2\pi} |\Delta_{k(r)}(x)|^p dx\right)^{1/p} |E_r|^{1/q} \leqq 3^{1/p}\delta_r k_r^{-1}.$$

By Lemma 3, there is a set F_r of measure not more than $A\delta_r^{1/2}\log\lambda_r$ for some constant A such that for x in F_r'

$$(4) \qquad \sup_{n\leqq\lambda_r^{2k_r}} \left|s_n(x;\, \mathfrak{X}\Delta_{k(r)})\right| \leqq \delta_r^{1/2}.$$

By (3), the square of the L^2 norm of the function $(1-\mathfrak{X})\Delta_{k(r)}$ does not exceed $3k_r^{\gamma-1}\delta_r$. We apply to this function the methods of [1]. Let its Fourier series be $\sum_{n=-\infty}^{+\infty} d_ne^{inx}$; let $L_r=\langle(n_{k(r)+1}-n_{k(r)})/k_r^{\gamma}\rangle$, $J_r=\langle k_r^{\gamma}\rangle-1$; and let

$$\mathcal{E}_\mu(x) = \sum_{|n|=n_{k(r)}+1+\mu L_r}^{n_{k(r)}+(\mu+1)L_r} d_ne^{inx}, \qquad \mu = 0, 1, \cdots, J_r.$$

Since

$$\sum_{\mu=0}^{J_r}\int_0^{2\pi} |\mathcal{E}_\mu(x)|^2 dx \leqq \int_0^{2\pi} |(1-\mathfrak{X}(x))\Delta_{k(r)}(x)|^2 dx \leqq 3k_r^{\gamma-1}\delta_r$$

then for at least one μ, say $\mu(r)$, $0\leqq\mu(r)<\langle k_r^{\gamma}\rangle$,

$$(5) \qquad \int_0^{2\pi} |\mathcal{E}_{\mu(r)}(x)|^2 dx \leqq Ck_r^{-1}\delta_r$$

for some constant C. Combining this result with Lemma 2, we obtain

$$\int_0^{2\pi} \sup_{n \leq \lambda_r^{2k_r}} \left| s_n(x; \mathcal{E}_{\mu(r)}) \right|^2 dx \leqq 2CA_2\delta_r(\log \lambda_r). \tag{6}$$

Let $N_r = n_{k(r)} + \mu(r)L_r$. Let the sequence $\{m_\nu\}$ take on the values m such that $N_r \leqq m \leqq N_r + L_r$, $r = 1, 2, \cdots$. For any such m there is an r such that

$$s_m(x; f) = s_{N_r}(x; f) + \{s_m(x; \Delta_{k(r)}) - s_{N_r}(x; \Delta_{k(r)})\}. \tag{7}$$

Now we write

$$\begin{aligned} s_m(x; \Delta_{k(r)}) - s_{N_r}(x; \Delta_{k(r)}) = &\{s_m(x; \mathfrak{X}\Delta_{k(r)}) - s_{N_r}(x; \mathfrak{X}\Delta_{k(r)})\} \\ &+ \{s_m(x; (1 - \mathfrak{X})\Delta_{k(r)}) - s_{N_r}(x; (1 - \mathfrak{X})\Delta_{k(r)})\}. \end{aligned} \tag{8}$$

By (4), the first bracketed term on the right of (8) does not exceed in absolute value $2\delta_r^{1/2}$ outside a set of measure not more than A $(\log \lambda_r)\delta_r^{1/2}$. The second bracketed term on the right of (8) is $s_m(x; \mathcal{E}_{\mu(r)})$, which, by virtue of (6), does not exceed in absolute value $\delta_r^{1/4}$ outside a set of measure not more than $2CA_2$ $(\log \lambda_r)\delta_r^{1/2}$. Thus

$$\sup_{N_r \leqq m \leqq N_r + L_r} \left| s_m(x; \Delta_{k(r)}) - s_{N_r}(x; \Delta_{k(r)}) \right| \leqq 2\delta_r^{1/2} + \delta_r^{1/4}, \tag{9}$$

for x in a set G_r' where $|G_r| \leqq B$ $(\log \lambda_r)\delta_r^{1/2}$ for some constant B. From our previous comments about the smallness of δ_r, it follows that choices of $\{\lambda_r\}$ and $\{k_r\}$ can be made, even if λ_r is allowed to increase to ∞ slowly enough, so that $\sum_{r=1}^{\infty} |G_r| < \infty$, i.e. so that almost every x belongs to all sets G_r' for all sufficiently large r. The sequence $\{N_r\}$ can be made lacunary so that $s_{N_r}(x; f)$ converges to f almost everywhere. From (7) and (9) we deduce that $s_{m_\nu}(x; f)$ converges to f almost everywhere.

Since $\lambda_r^{k_r} \leqq N_r + L_r \leqq \lambda_r^{k(r)+1}$, we have for the sequence $\{m_\nu\}$

$$\begin{aligned} (\log (N_r + L_r))^\gamma \frac{\sigma(N_r + L_r)}{N_r + L_r} &\geqq k_r^\gamma (\log \lambda_r)^\gamma \frac{L_r}{N_r + L_r} \\ &\geqq (\log \lambda_r)^\gamma \left(1 - \frac{1}{\lambda_r}\right) - \frac{k_r^\gamma (\log \lambda_r)^\gamma}{\lambda_r^{k(r)+1}}. \end{aligned}$$

The second term on the right goes to 0, and the first term can be made larger than 1 so that (C_p) is satisfied. The limit can, in fact, be made infinite if $\gamma > 0$, i.e. if $1 < p < 2$; but then the analogy with the L^2 theorem is lost.

It remains only to get rid of the restriction that $c_n = 0$ for $n \leqq 0$. Thus we may write $f(x) = f_1(x) + f_2(-x)$ where f_i belongs to L^p and where the Fourier coefficients of negative index for the two functions $f_1(x)$ and $f_2(x)$ are 0. We may proceed as before to find an integer $k(r)$ such that inequalities analogous to (3) hold simultaneously for both of these functions; and then a single integer $\mu(r)$ such that inequalities analogous to (5) hold for both. Hence a single sequence $\{m_\nu\}$ satisfying (C_p) is constructed so that $s_{m_\nu}(x;f_i)$ converges almost everywhere to f_i, $i=1, 2$. A generalization of this technique leads to the following generalization of Theorem 1: *given a sequence* $\{f_n\}$ *of functions in* L^p, $1<p\leqq 2$, *there is a sequence of integers satisfying* (C_p) *such that almost everywhere* $s_{m_\nu}(x;f_n)$ *converges to* f_n *for every* n. The same principle applies to all our theorems, and we do not mention it again.

With respect to a lacunary sequence $\{n_k\}$ we shall say that the sequence $\{m_\nu\}$ of positive integers satisfies the condition (c_γ) if in every block (n_k, n_{k+1}) there is a block of terms from $\{m_\nu\}$ of length at least

$$(c_\gamma) \qquad \left\langle \frac{n_{k+1}-n_k}{(\log n_{k+1})^\gamma} \right\rangle.$$

The next theorem is stated in terms of the (c_γ) condition, and it merges with the corresponding L^2 theorem except for minor adjustments [1, p. 392].

THEOREM 2. *Let f belong to L^p, $1<p<2$; let $\gamma>3q/2-2$ where q is the conjugate of p; and let $\{n_k\}$ be a lacunary sequence. Then there is a sequence $\{m_\nu\}$ satisfying (c_γ) with respect to $\{n_k\}$ such that $s_{m_\nu}(x;f)$ converges to f almost everywhere.*

In order to make use of the Littlewood-Paley result, we must have a lacunary sequence $\{N_j\}$ such that N_{j+1}/N_j is bounded above. There is no difficulty in imbedding the sequence $\{n_k\}$ in a sequence $\{N_j\}$ satisfying the condition $1<\alpha\leqq N_{j+1}/N_j\leqq\beta$. This can be done, for example, by adjoining to $\{n_k\}$ the appropriate terms of a geometric progression. If the theorem is proved for the sequence $\{N_j\}$, and if $n_{k+1}=N_{j+1}$, then

$$\left\langle \frac{N_{j+1}-N_j}{(\log N_{j+1})^\gamma} \right\rangle \geqq \left\langle \frac{n_{k+1}(1-\alpha^{-1})}{(\log n_{k+1})^\gamma} \right\rangle \geqq \frac{1-\alpha^{-1}}{2} \left\langle \frac{n_{k+1}-n_k}{(\log n_{k+1})^\gamma} \right\rangle$$

for n_k big enough. Hence, apart from the constant factor $(1-\alpha^{-1})/2$, the sequence $\{m_\nu\}$ satisfies (c_γ) with respect to $\{n_k\}$. As our proof will show, compensation can be made for this factor. Thus we begin by assuming that $1<\alpha\leqq n_{k+1}/n_k\leqq\beta$. In fact we shall assume that $n_k=3^k$ and, as before, that $c_n=0$ for $n\leqq 0$. The adjustments in the proof for the general case are quite minor. $\Delta_k(x)$ will have the same meaning as previously. By Hölder's inequality

$$\sum_{k=1}^{\infty} k^{-(\gamma-q+1)(p-1)} \left| \Delta_k(x) \right|^p \leqq \left(\sum_{k=1}^{\infty} \left| \Delta_k(x) \right|^2 \right)^{p/2} \left(\sum_{k=1}^{\infty} k^{-\tau} \right)^{1-p/2}$$

where $\tau=2(\gamma-q+1)(p-1)/(2-p)$. The second factor of the right side is a convergent series since $\tau>1$. Integrating this inequality gives

$$\sum_{k=1}^{\infty} \delta_k = \sum_{k=1}^{\infty} k^{-(\gamma-q+1)(p-1)} \int_0^{2\pi} |\Delta_k(x)|^p dx < \infty$$

by (1). Let E_k be the set of x values for which $|\Delta_k(x)| > k^{\gamma}$. On E_k, $|\Delta_k(x)| < k^{-\gamma(p-1)}|\Delta_k(x)|^p$. Let $\mathfrak{X}_k(x)$ be the characteristic function of E_k. Then

$$\int_0^{2\pi} |\mathfrak{X}_k(x)\Delta_k(x)| dx \leqq k^{-\gamma(p-1)} \int_0^{2\pi} |\Delta_k(x)|^p dx = k^{-1}\delta_k.$$

Hence, by Lemma 3, if $F_{k,M}$ is the set where $\sup_{n\leq 3^{k+1}} |s_n(x; \Delta_k\mathfrak{X}_k)| > M$, then

$$(10) \qquad |F_{k,M}| \leqq \frac{A\delta_k}{M}$$

for some constant A. For E_k', the complement of E_k, we have as before

$$(11) \qquad \int_{E_k'} |\Delta_k(x)|^2 dx \leqq k^{\gamma(2-p)} \int_0^{2\pi} |\Delta_k(x)|^p dx = k^{\gamma-1}\delta_k.$$

Let d_n be the nth Fourier coefficient of $\Delta_k(1-\mathfrak{X}_k)$, let $L_k = \langle 2(3)^k/k^{\gamma}\rangle$, and let

$$\mathcal{E}_j(x) = \sum_{|n|=3^k+1+jL_k}^{3^k+(j+1)L_k} d_n e^{inx}, \qquad j = 0, 1, \cdots, \langle k^{\gamma}\rangle - 1 = J.$$

Now by (11)

$$\sum_{j=0}^{J} \int_0^{2\pi} |\mathcal{E}_j(x)|^2 dx \leqq \int_0^{2\pi} |\Delta_k(x)|^2 (1 - \mathfrak{X}_k(x)) dx \leqq k^{\gamma-1}\delta_k.$$

For at least one j in the given range,

$$\int_0^{2\pi} |\mathcal{E}_j(x)|^2 dx \leqq 2k^{-1}\delta_k$$

since there are $J+1 = \langle k^{\gamma}\rangle$ numbers $\int_0^{2\pi} |\mathcal{E}_j(x)|^2 dx$. Denote a suitable j by $j(k)$. Lemma 2 implies

$$(12) \qquad \int_0^{2\pi} \sup_{n\leqq 3^{k+1}} |s_n(x; \mathcal{E}_{j(k)})|^2 dx \leqq 2A_2 \frac{k+1}{k} (\log 3)\delta_k.$$

Now we let $\{m_\nu\}$ take on the values m such that $N_k = 3^k + j(k)L_k \leqq m \leqq 3^k + [j(k)+1]L_k$, $k = 1, 2, \cdots$. The sequence $\{m_\nu\}$ satisfies (c_γ). For any such m, there is a k such that

$$(13) \qquad \begin{aligned} s_m(x; f) &= s_{N_k}(x; f) + \{s_m(x; \Delta_k) - s_{N_k}(x; \Delta_k)\} \\ &= s_{N_k}(x; f) + \{s_m(x; \Delta_k\mathfrak{X}_k) - s_{N_k}(x; \Delta_k\mathfrak{X}_k)\} + s_m(x; \mathcal{E}_{j(k)}). \end{aligned}$$

Let $\{\mu_k\}$ be a sequence increasing to ∞ slowly enough so that $\sum_{k=1}^{\infty} \mu_k^2 \delta_k < \infty$. From (10) and (12) it follows that

$$\sup_{N_k \leqq m \leqq N_k+L_k} \left| \left\{ s_m(x; \Delta_k \mathfrak{X}_k) - s_{N_k}(x; \Delta_k \mathfrak{X}_k) \right\} + s_m(x; \mathcal{E}_{j(k)}) \right| \leqq \frac{1}{\mu_k}$$

outside a set G_k of measure not exceeding $8A_2[(k+1)/k]$ (log $3)\delta_k\mu_k^2+4A\,\delta_k\mu_k$. Thus $\sum_{k=1}^{\infty} |G_k| < \infty$ so that almost everywhere the above is true for all sufficiently large k. The sequences $\{N_k\}$, k odd and k even, are separately lacunary so that $s_{N_k}(x; f)$ converges to f almost everywhere. From (13) we deduce that $s_{m_\nu}(x; f)$ converges to f almost everywhere.

4. The tools used in the proofs of Theorems 1 and 2 are also available for Walsh series, at least for the L^2 case [5], but not for general orthonormal series. In fact the construction used in the proof of a theorem of Menchoff [3, p. 167] can be modified so as to show the following: *there is an orthonormal system $\{\phi_n\}$ and an L^2 function f such that for any sequence $\{m_\nu\}$ of upper density one, the sequence of partial sums of index m_ν of the $\{\phi_n\}$ expansion of f diverges almost everywhere.*

Our theorems do have analogues for the case of Fourier integrals. We give a sample in the following theorm. Let

$$S_\omega(x;f) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(t) \frac{\sin \omega(x-t)}{x-t} dt$$

which is defined for all x and ω if f belongs to $L(-\infty, \infty)$.

THEOREM 3. *If f belongs to $L(-\infty, \infty)$, and if for some p, $1<p\leqq 2$, f belongs to L^p over any finite interval, then there is a sequence $\{m_\nu\}$ of positive integers satisfying* (C_p) *such that $S_{m_\nu}(x;f)$ converges to f almost everywhere.*

For $n=0, \pm 1, \pm 2, \cdots$, we define $f_n(x)=f(x+2\pi n)$, $0\leqq x<2\pi$, and by periodicity elsewhere. Since each f_n belongs to $L^p(0, 2\pi)$, there is according to our remarks following the proof of Theorem 1 a sequence $\{m_\nu\}$ satisfying (C_p) such that almost everywhere $s_{m_\nu}(x; f_n)$ converges to f_n for each n. Our theorem then follows from the equiconvergence principle [8, p. 306] which implies that $S_{m_\nu}(x;f)-s_{m_\nu}(x;f_n)$ converges to 0 for $2\pi n<x<2\pi(n+1)$.

The techniques of the proof of Theorem 1 can also be used to find a sequence $\{m_\nu\}$ satisfying (C_p), corresponding to a function f satisfying a certain continuity condition, such that the order of convergence of $s_{m_\nu}(x;f)$ to f reflects this continuity. Let

$$\omega_p(\delta;f) = \sup_{0<h\leqq\delta} \left[\int_0^{2\pi} |f(x+h)-f(x)|^p dx \right]^{1/p}.$$

THEOREM 4. *If f belongs to L^p, $1<p\leqq 2$, and if $\omega_p(\delta; f)=o[(\log 1/\delta)^{-\alpha}]$, $\alpha\geqq 0$, then there is a sequence $\{m_\nu\}$ satisfying* (C_p) *such that almost everywhere $s_{m_\nu}(x; f)-f(x)=o[(\log m_\nu)^{-\alpha}]$.*

Let $\sigma_n(x; f)$ be the nth Cèsaro mean of the Fourier series of f at x. We have [8, p. 85]

$$\int_{-\pi}^{\pi} | \sigma_n(x;f) - f(x) |^p dx \leqq \frac{1}{\pi} \int_{-\pi}^{\pi} K_n(t) dt \int_{-\pi}^{\pi} | f(x+t) - f(x) |^p dx$$

where K_n is the nth Fejer kernel. The integration along the t axis can be split into two parts: $|t| \leqq n^{-1/2}$ and $n^{-1/2} < |t| \leqq \pi$. If use is made of the inequality $K_n(t) \leqq \pi^2(n+1)^{-1}t^{-2}$ for the second part, it follows that

$$\int_{-\pi}^{\pi} | \sigma_n(x;f) - f(x) |^p dx = o[(\log n)^{-\alpha p}].$$

Then [8, p. 153] if we let $f(x) = f_n(x) + \sigma_n(x;f)$

$$\mathfrak{M}_p[f - s_n] = \left[\int_0^{2\pi} | f(x) - s_n(x;f) |^p dx \right]^{1/p} \leqq A_p \mathfrak{M}_p[f_{n-1}]$$

for some constant A_p. Now we use the same notation as in the proof of Theorem 1 except for the following minor change: $\Delta_k(x) = \sum_{|n|=n_k+1}^{n_{k+1}} c_n e^{inx}$. We may write $\delta_r = k_r^{-\alpha p}\delta_{r,1}$ where $\delta_{r,1} = o(1)$ as r goes to ∞, from what has been shown about the mean convergence of $s_n(x;f)$. E_r is now defined as the set of x values for which

$$\sum_{k=k_r}^{2k_r-1} |\Delta_k(x)|^2 > k_r^{1/(p-1)-2\alpha}.$$

In the same way as before, $k(r)$ is found such that

$$\int_{E_r'} | \Delta_{k(r)}(x) |^2 dx \leqq 3k_r^{\gamma-2\alpha-1}\delta_{r,1}, \qquad \int_0^{2\pi} | \Delta_{k(r)}(x) |^p dx \leqq 3k_r^{-p/2-\alpha p}\delta_{r,1}.$$

Thus there is a set F_r with measure not exceeding $A\delta_{r,1}^{1/2}(\log \lambda_r)^{\alpha+1}$ such that for x in F_r'

$$\sup_{n \leqq \lambda_r^{2k_r}} | s_n(x; \mathfrak{X}\Delta_{k(r)}) | \leqq \delta_{r,1}^{1/2} k_r^{-\alpha}(2 \log \lambda_r)^{-\alpha}$$

L_r and $\mathcal{E}_\mu$ have their previous meaning. Analogous to (6) is the following:

$$\int_0^{2\pi} \sup_{n \leqq \lambda_r^{2k_r}} | s_n(x; \mathcal{E}_{\mu(r)}) |^2 dx \leqq CA_2(\log \lambda_r)\delta_{r,1}k_r^{-2\alpha}.$$

That is, $\sup_{n \leqq \lambda_r^{2k_r}} | s_n(x; \mathcal{E}_{\mu(r)}) | \leqq \delta_r^{1/2}(2k_r \log \lambda_r)^{-\alpha}$ outside a set of small measure. Since

$$\int_0^{2\pi} | s_{N_r}(x;f) - f(x) |^p dx = o(k_r^{-\alpha p})$$

we have, for some small δ, $| s_{N_r}(x;f) - f(x) | \leqq \delta(2k_r \log \lambda_r)^{-\alpha}$ outside a set of small measure. Now the proof can be completed by our original method.

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